De Morgan's De Morgan's Laws

Duality in the Emergence of Formal Logic

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1 Introduction

The first idea that comes to our minds when we think about duality *in logic* is, without any doubt, the duality expressed by the so-called *De Morgan's laws*. Namely:

$$\begin{aligned} \neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi \\ \neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi \end{aligned} \tag{1}$$

As it is well-known, these laws express the exchange of the logical operations of conjunction and disjunction ($\wedge$ and $\vee$) as the result of the interaction between the negation ($\neg$) of a formula and that of its terms, within the propositional fragment of classical logic.

In their current form, De Morgan's laws are typically understood in terms of binary propositional connectives truth-conditionally defined. From this point of view, once a formal system for propositional logic has been laid out and associated to a truth-functional semantics, the duality between the conjunctive and disjunctive connectives with respect to negation—also defined as a (unary) truth-functional connective—follows trivially from the definitions, by simple combinatorial manipulation. In other terms, within such a setting, the expression of such properties in (1) constitutes tautological propositions: whatever the assignment of truth values for the propositional variables ϕ and ψ, both propositions turn out to be true, by virtue of the truth-functional definition of its connectives. Hence their status as logical *laws*.

Conjunction and disjunction are not the only dual operations in modern logic. The universal and existential quantifier, the modal operators of necessity and possibility in modal logic, or the operators of obligation and permission in deontic logic are but a few examples of logical operators exhibiting dual properties. Significantly, all these cases share a common structure, which Demey and Smessaert characterize as involving a certain "interaction between an 'external' and an 'internal' negation of some kind" (Demey and Smessaert, 2016, 1).

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Following these authors, such a property can be formally expressed as follows: given two Boolean Algebras $\mathbb{A} = \langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, \neg_{\mathbb{A}}, \top_{\mathbb{A}}, \bot_{\mathbb{A}} \rangle$ and $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \bot_{\mathbb{B}} \rangle$, and n-ary operators $O_1, O_2 : \mathbb{A}^n \to \mathbb{B}$, the operators O_1, O_2 are dual if and only if (Demey and Smessaert, 2016, 14):

$$\forall a_1, \ldots, a_n \in A : O_1(a_1, \ldots, a_n) = \neg_{\mathbb{B}} O_2(\neg_{\mathbb{A}} a_1, \ldots, \neg_{\mathbb{A}} a_n) \tag{2}$$

Based on this general characterization, it is easy to see that when $\mathbb{A} = \mathbb{B}$ and O_1, O_2 are respectively the operations of conjunction and disjunction, we recover De Morgan's laws.

Despite this general perspective from which the laws attributed to De Morgan appear only as a special case, the latter remain the paradigmatic form of *logical* duality. In particular, De Morgan's laws have played a decisive role in the history of logic, orienting both the discovery of new dualities and the elaboration of a general concept and definition of duality in logic as the one just given.

From a historical viewpoint, the exchange of conjunction and disjunction through negation was known within the logical tradition long before Augustus De Morgan's work in the mid-19th century. Bonevac and Dever, for instance, mention an explicit consideration of this phenomenon by William of Ockham in the early 14th century (2012, 196), also suggesting that the Stoics already understood the famous laws (2012, 182). The importance of De Morgan's own statement of those duality properties lies, however, in that it takes place as part of one of the first attempts to mathematize the logic of propositions, anticipating, if not already inaugurating our current approach to formal propositional logic.

Grattan-Guinness points to De Morgan's 1858 "On the Syllogism, No. III, and on Logic in general" (De Morgan, 2019) as the place where De Morgan states the laws named after him (Grattan-Guinness, 2000, 36). Indeed, in this paper, De Morgan affirms that:

> The contrary of an aggregate is the compound of the contraries of the aggregants: the contrary of a compound is the aggregate of the contraries of the components. Thus (A, B) and AB have ab and (a, b) for contraries. (De Morgan, 2019, 119)

Aggregates and compounds refer here to the classes of individuals corresponding to names formed from other names through aggregation (extension) or composition (intension), respectively (De Morgan, 2019, 118). However, this is not De Morgan's first mention of such dual properties. Indeed, what can arguably count as the first explicit statement of De Morgan's laws in his own work can be found on page 115 of his seminal *Formal Logic*, published more than a decade before, in 1847 (De Morgan, 2014, 115-116):

> P, Q, R, being certain names, if we wish to give a name to everything which is all three, we may join them thus, PQR: if we wish to give a name to every thing which is either of the three (one or more of them) we may write P,Q,R: if we want to signify any thing that is either both P and Q, or R, we have PQ,R. The contrary of PQR is p,q,r; that of P,Q,R is pqr; that of PQ,R is (p,q)r: in contraries, conjunction and disjunction change places. (De Morgan, 2014, 115-116)

Somewhat surprisingly, De Morgan provides what looks like a sheer repetition of the same statement only two pages later:

> The contrary of PQ, is p,q; that of P,Q is pq. *Not both* is either not one or not the other, or not either. *Not either P nor Q* (which we might denote by :P,Q or .P,Q) is logically '*not P and not Q*' or pq: and this is then the contrary of P,Q. (De Morgan, 2014, 118)

Grattan-Guinness is not unaware of these pages, to which he refers in different occasions. But the fact that he finds them "rather unclear" (Grattan-Guinness, 2000, 30) might explain why he prefers De Morgan's later formulation, where it is easier to resort to an interpretation of the laws in terms of classes and class operations, closer to our current understanding.

Yet, from a historical perspective, it might be of interest to go back to De Morgan's early logical work and try to reconstruct the sense of the framework and ideas motivating the role of duality at the moment of emergence of mathematized logic. For one thing is clear: our current understanding of logical duality through De Morgan laws reverses the direction of the historical development of logical thought, where the statement of dual principles by De Morgan largely precedes the establishment of propositional logic as we know it. Their historical and logical meaning is, therefore, not immediately reducible to our contemporary account.

The following pages are then an attempt to understand the role of duality in a logical framework where the elementary intuitions of propositional logic are unavailable, given that the lack of the latter does not prevent dual properties from being stated as logical principles. Substantially conceived as a symbolic system, such a framework encouraged all sorts of symmetries, which Grattan-Guinness characterized as "rather akin to duality" (Grattan-Guinness, 2000, 36). In this sense, these pages can be seen as an effort to elucidate what "akin to duality" means[1] and how this circumstance relates to De Morgan's own "rather unclear" statement of his famous laws.

To that end, we will investigate in De Morgan's early work the two main features involved in the formal characterization of duality given above, namely: 1) the problem of negation, and more precisely, the interplay between an internal and an external negation; and 2) the operations of conjunction and disjunction with respect to that interplay.

In an attempt to grasp the coherence of De Morgan's thought, we will propose a contemporary reconstruction of some aspects of his system with the aid of elementary set-theoretical tools. Nevertheless, this reconstruction should not be seen as a formalization of his system, but as a heuristic device revealing, if anything, that a contemporary reconstruction intending to remain faithful to De Morgan's original system is far from straightforward.

From this analysis, it will appear that, historically, far from being a more or less trivial consequence of the development of a truth-functional propositional setting, dual properties emerged in logic as the expression of an embryonic yet sophisticated class-theoretical semantics, enriching the traditional treatment of

[1] Grattan-Guinness attributes the dual properties of De Morgan's system specifically to *notations*: "Some of these [De Morgan's] collections of notations displayed duality properties, although De Morgan did not emphasise the feature" (Grattan-Guinness, 2000, 36). We will see that duality is rooted in symbolic principles that go beyond simple notation.

syllogistic propositions with an underlying sub-propositional ("nominal") dimension. What is more, we will see that the complex articulation between the nominal and the propositional levels suggests an original conception of the formalization of logic where propositional connectives may emerge from dual behaviors rather than the other way round.

2 The Many Classes of Negation

In this section, we will attempt to reconstruct De Morgan's views on negation in his early logical work, stemming from his introduction of *contraries*, which he considered as one of the his main originalities (De Morgan and De Morgan, 2010, 157). However, logical negation cannot be reduced to simple contrariness in his system. We will see that negation adopts different forms, resulting from De Morgan's sophisticated reworking of the received syllogistic framework, and more precisely, from his analysis of syllogistic propositions and his progressive elaboration of class semantics based on names, which allowed him to perform successive extensions of the syllogistic propositional setting.

2.1 The Analysis of Logical (Pr)oppositions

In the most general terms, De Morgan's logical endeavor can be characterized as the attempt to mobilize the symbolical power of the nascent British abstract algebra to analyze the received syllogistic framework and provide a new basis for logical thought, capable, among others, of enlarging the inferential principles of logic.[2] Accordingly, the horizon of De Morgan's algebraization of logic is outlined by the singular way in which he conceives the destiny of that emerging discipline of mathematics:

> The progress of algebra as distinguished from arithmetic, is marked by the gradual approach to the following theorem, that *every pair of opposite relations is undistinguishable from every other pair, in the instruments of operation which are required.* (De Morgan, 2019, 23)

Irrespective of its veracity concerning the evolution of abstract algebra, this statement can be taken as condensing De Morgan's overall logical enterprise. Thus, throughout his logical work, De Morgan will have recourse to profuse

[2] It has been said that De Morgan's particular use of algebraic symbolism disregarded the "operational aspect of logic as calculus" (Peckhaus, 2009, 169). This could indeed be the impression his system conveys at first sight, if compared to the explicit symbolization of logical operators one can find in the contemporary work of his colleague and friend George Boole. De Morgan explicitly points out the difference in their use of algebra in a letter to Boole in November 1847: "There are some remarkable similarities between us. Not that I have used the connexion of algebraical laws with those of thought, but that I have employed mechanical modes of making transitions, with a notation which represents our head work." (Boole, 1982, 25). However, while it is true that De Morgan's algebraization resides in the proliferating use of algebraic notation rather than in the explicit definition of operators out of combinatorial laws, operational aspects are not entirely absent from De Morgan's proposed system of logic. They emerge naturally from his algebraic symbolization of the classic syllogistic setting. In his own terms, the application of algebraic symbols to logic has a tendency "to develop [...] the *algebra* of the laws of thought." (De Morgan, 2019, 22). For a particularly remarkable example of operational aspects of his system, see footnote 16 below.

algebraic symbolizations, mobilizing what can be understood as a unique figure, of *opposition* of algebraic origin, operating at different levels.[3]

In more than one sense, the logical role of such a figure can be understood as that of negation. And as we will see, in its most prominent cases, the recourse to opposition can indeed be interpreted as class complementation. However, it is important not to project the modern idea of propositional negation too quickly upon the operation of complementation. For De Morgan uses complementation over very different kinds of classes, and the question of knowing which one should be interpreted as propositional negation is not trivial. Nevertheless, one thing seems clear: a significant part of De Morgan's reworking of logic takes the form of an exploration of the semantics of logical negation, determined through the interplay between all the forms of logical opposition stemming from the multiple uses of class complementation. Thus, negation will alternately take the form of *contrariness*, *contranominality*, and *contradictory denial*, depending on the action of opposition respectively over names, propositional relations, and propositional forms, following a practice of complementation over the corresponding yet non necessarily explicit classes.

2.2 Putting Syllogistics in Order

While the mathematical background of De Morgan's logical work is given by British abstract algebra, from a logical standpoint, his enterprise must be understood against the backdrop of syllogistics, which experienced a renewed interest in the early 19th century through the works of authors like Whately, Hamilton or Bentham (cf. Evra, 2000). It is this syllogistic framework that De Morgan seeks to extend by having recourse to the fresh resources provided by the new symbolic algebra, even if—and to some extent precisely because—such an extension should lead logic outside its traditional syllogistic constraints. De Morgan's main strategy will then be to interpret the multiple forms of traditional syllogistic oppositions in algebraic terms.

Within syllogistics, opposite relations were customarily shown to hold between *propositions* that differ either in *quantity* (universal or particular) or *quality* (affirmative or negative). The possible combinations of those two propositional dimensions provide the four primary forms of syllogistic propositions: 'Every X is Y', 'No X is Y', 'Some X is Y' and 'Some X is not Y', traditionally symbolized by the letters A, E, I and O respectively. The different opposite relations between those forms—namely: contrariety, contradiction, subcontrariety, and subalternation—are then typically presented in the schema that became known as the "square of oppositions" (see fig. 1.).[4]

[3]This orientation will become increasingly explicit after his *Formal Logic*, starting with his 1850 article "On the Syllogism II" (reprinted in De Morgan, 2019). Cf., for instance, (2019, 23): "The suggestions of symbolic notation have led me to more recognition than is usually made of harmonies which exist among various pairs of opponent notions common in logical thought.", and again, some pages later (2019, 26): "I think it reasonably probable that the advance of symbolic logic will lead to a calculus of opposite relations, for mere inference, as general as that of + and − in algebra." For a detailed analysis of De Morgan's intense notational practice related to this orientation in his later work, see Heinemann (2018).

[4]A systematic presentation of such opposite relations as of the beginning of the 19th century can be found in Whately's *Elements of Logic* Whately (1871), a work first appeared in 1826 as an article in *Encyclopaedia Metropolitana*, credited by De Morgan as the source of the restoration of logical study in England (De Morgan, 2019, 247).

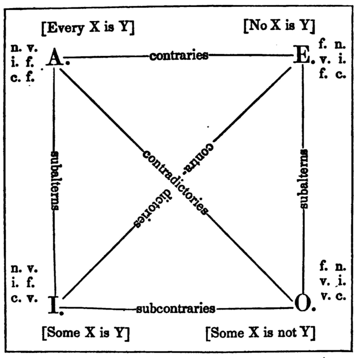

Figure 1: Whately's "scheme" of propositional oppositions (Whately, 1871, 78)

Propositional oppositions within syllogistics are thus intimately related to the internal structure of the proposition itself, classically restricted to quantity and quality. Note that, among all those oppositions, *contradiction exhibits a dual behavior* with respect to these two dimensions. Indeed, the values of both quantity and quality are exchanged in contradictory propositions.

The first step in De Morgan's early logical construction is then to enrich the analysis of the structure of the proposition so that novel forms of (oppositional) relations can be grasped between propositions. Thus, in addition to quantity and quality, De Morgan will introduce a dimension of *order* between the terms as well as of *contrariness* for each of the two terms (subject and predicate). The introduction of contrariness constitutes the cornerstone for his novel treatment of logical negation. However, a glimpse at his approach to order can already give us an idea of the general strategy adopted in his work.

With "order" De Morgan refers to the permutation of terms that takes place in inverse propositions, such as 'Every X is Y' and 'Every Y is X' (De Morgan, 2014, 56). The inversion of terms in a proposition was a well-known operation in syllogistics. Nevertheless, order was not usually considered an internal dimension of the proposition itself, and the task of establishing the equivalence of inverse propositional forms motivated a theory of "conversion" lying outside the theory of oppositions.

By including order as yet another dimension of the propositional structure, De Morgan doubles the number of elementary propositions, which goes from four (i.e., A, E, I, and O) to eight. But instead of denoting the four new forms with new characters, he introduced a novel notation that made the internal structure of propositions visible. Thus, A, E, I, and O become respectively expressed by X)Y, X.Y, XY and X:Y. Interestingly, De Morgan displays his new notation in a way analogous to the traditional square, although all relations between the original propositions have been now replaced by a simple vertical line dividing propositions into two columns, announcing a new kind of oppositional organization to come (fig. 2).

Thanks to this new notation, the four new forms resulting from considering the order of the terms can now be expressed by simply exchanging the places of X and Y, namely: Y)X, Y.X, YX and Y:X. Of the resulting eight elementary forms or modes, X.Y and XY were known (through the traditional theory of conversion) to be equivalent to their corresponding inverses from the viewpoint

X)Y means 'Every X is Y'	**X.Y means 'No X is Y'**
X:Y — 'Some Xs are not Ys'	**XY — 'Some Xs are Ys'**

Figure 2: De Morgan's notation for syllogistic propositions (De Morgan, 2014, 60)

of inference (i.e., they are "convertibles").[5] We are then left with only six elementary cases, which, as De Morgan points out, can be arranged by contradictories (De Morgan, 2019, 4), namely:

X)Y	is the contradictory of	X:Y
Y)X	"	Y:X
X.Y ou Y.X	"	XY ou YX

This first extension of the classical propositional framework of syllogistics is hardly significant since it only rearranges the received syllogistic material without introducing any new interesting logical property. As such, it constitutes only the first step into a more elaborate construction, including further extensions of the propositional setting by which new properties will be revealed. However, it already reveals the strategy of De Morgan's analysis: *by enriching the internal structure of the proposition* (in this case, adding order to quantity and quality as propositional dimensions), *De Morgan is able to display an overall relation of opposition between the propositions themselves, based on contradiction.* If such oppositional relation were to be operationalized and symbolized in some way (say, as a kind of propositional negation), then it is easy to see how the exchange of values along the propositional dimensions (e.g., universal and particular in the quantity dimension), to which De Morgan has given independent status through his novel notation (dots and parenthesis), would interact with that operation in a way where dual mechanisms are likely to become prominent.

This initial glimpse at De Morgan's strategy provides a first intuition of how the characteristics of his system are "akin to duality." Certainly, an operation of negation like the one involved in logical duality is far from being easily associated with any of the system's existing components in its current state. However, we should note this interplay between two different levels, namely a propositional and a sub-propositional one, where oppositional behaviors in the former (e.g., contradiction) can be seen to result from the interaction between components of the latter (e.g., quantity, quality and order).

2.3 From syllogistic logic to the logic of names...

The first original form of negation in De Morgan's edifice stems from his introduction of a notion of *contrariness*. With it, a semantics of classes will start to be introduced in the propositional setting. Indeed, for De Morgan, contrariness will invariably be associated with some form of class complementation and, as such, expected to exhibit dual behaviors with respect to relations between classes. However, this does not mean that a complete theory of classes featuring well-defined class operators such as union or intersection becomes fully available

[5] The choice of dot notation and simple juxtaposition is explicitly made by De Morgan to suggest the commutative behavior of such notation in algebra (cf. De Morgan, 2019, 4).

to interpret the operations on logical objects. In De Morgan's work, relations between classes are not given as primitives and will become identifiable only as part of the logical interpretation resulting from the introduction of contrariness in the syllogistic setting.

Accordingly, we must be careful not to understand contrariness in terms of contemporary propositional negation. First, without a full class semantics, contrariness cannot be seen here as an elementary logical operator together with conjunction and disjunction, for instance. For now, contrariness should instead appear as yet another dimension of the proposition, together with quantity, quality, and order. But more importantly, contrariness does not concern propositional units but rather sub-propositional ones, namely *terms*.

The terms of a proposition like "Every X is Y" (or "X)Y" in De Morgan's notation) are what the X and the Y stand for. Here again, we should resist the immediate identification of terms with classes. De Morgan will interpret such terms as *names*, hence, as *linguistic* units, and more precisely, as words: "A term, or name, is merely the word which it is lawful to apply to any one of a collection of objects of thought" (De Morgan, 2019, 1).

The centrality of names in De Morgan's logic can not be exaggerated. To such an extent that names constitute the sole object of the entire project of a formal logic as he conceives it:

> In all assertions, however, it is to be noted, once for all, that *formal logic*, the object of this treatise, deals with *names* and not with either the *ideas* or *things* to which these names belong. (De Morgan, 2014, 42, emphasis in the original)

The importance of this linguistic interpretation of logic resides in that, through names, logical terms can acquire a positive character, which differs from that of the ideas in the mind. Certainly, names are subjective, at least from a logical standpoint, for "they are the representations of the notion in the mind" (De Morgan, 2019, 2). Yet collections of objects provide an objective counterpart to those subjective entities, and the focus on names allows us to have a positive grasp on how to associate logical terms to this objective counterpart: if a name is clearly understood, then "of every object of thought we can distinctly say, this name does or does not, contain that object" (De Morgan, 2014, 37). Hence, for De Morgan, the act of naming, by which a name is attached to individual objects, constitutes a fundamental logical relation. To such an extent that, for all logical purposes, the meaning of copula *is* should be reduced to the identity of individual objects bearing different names:

> The following are the characteristics of the word *is* which, existing in any proposed meaning of it, make that meaning satisfy the requirements of logicians when they lay down the proposition 'A is B.' To make the statement distinct, let the proposition be doubly singular, or refer to one instance of each, one A and one B: let it be 'this one A is this one B.' (De Morgan, 2014, 50)

De Morgan's entire logical edifice will be ultimately built on the basis of this relation attributing names to individuals. The fecundity of this nominal approach to logic becomes manifest with the introduction of contrary names:

> In logic, it is desirable to consider names of inclusion with the corresponding names of exclusion: and this I intend to do to a much greater extent than is usual: inventing names of exclusion by the prefix not, as in tree and not-tree, man and not-man. Let these be called *contrary*,* or *contradictory*, names. [in note:* I intend to draw no distinction between these words.] (De Morgan, 2014, 37)

The significance of De Morgan's original theory of names can be better appreciated when compared to the difficulties witnessed by the Aristotelian tradition when dealing with the negation of terms. As De Morgan observes, from an Aristotelian point of view, contrary terms, such as 'not-man', are problematic since they are thought to be predicated of non-existent things. However, De Morgan's perspective overcomes this pitfall thanks to two original and interrelated interpretations of contrary terms.

The first one concerns his interpretation of the contrary of a name as what we would now call the complement of the class of objects representing its objective counterpart. Although this idea was not entirely new, the problems encountered in the Aristotelian tradition when trying to manipulate terms as complementary classes in an indefinite universe were one of the principal reasons behind the difficulties in providing a consistent theory of negation. De Morgan's solution to this problem is well-known: applied to names, contrariness always takes place within a particular reference frame: the *universe of discourse*, or universe of a proposition, as De Morgan calls it:

> [...] let us say that the whole idea under consideration is *the universe* (meaning merely the whole of which we are considering parts) and let names which have nothing in common, but which between them contain the whole idea under consideration, be called contraries *in, or with respect to, that universe*. (De Morgan, 2014, 38)

Faithful to his interpretation of terms as names, De Morgan relies on the behavior of names in "common language" when introducing the concept of the "universe of a proposition" (cf. De Morgan, 2014, 37). Yet, from the objective viewpoint of classes of objects,[6] universes guarantee that complementation is well defined in all cases, thus providing an appropriate semantics for contrariness of logical terms, overcoming the difficulties encountered by his predecessors. As De Morgan puts it:

> By not dwelling upon this power of making what we may properly (inventing a new technical name) call the *universe* of a proposition, or of a name, matter of express definition, all rules remaining the same, writers on logic deprive themselves of much useful illustration. And, more than this, they give an indefinite negative character to the contrary, as Aristotle did when he said that not-man was not the name of anything. (De Morgan, 2019, 2)

In this way, the terms, as they appear in a proposition, can now be understood as classes of objects within a definite universe, and their contraries consistently interpreted in terms of complementation over that universe.

[6]For De Morgan, universes are composed of objects and ideas alike (cf. De Morgan, 2014, ch. II). Since this difference is immaterial for the aims of this paper, we will continue to refer to them as objects (or individuals) in the rest of the paper.

The second originality concerning contrary terms comes from the strictly linguistic interpretation of terms as names. Indeed, De Morgan appeals to the functioning of names in natural language to show the relative character (or 'correlative', as he calls it) of contraries:

> ... *Briton* and *alien* are simple contraries; alien has no meaning of definition except not-Briton. But we cannot say that either term is positive or negative, except correlatively. As to a claim of right to be considered a prisoner of war, for instance, alien is the positive term, and Briton the negative one. We separate formal logic from language, if we refuse to admit this. (De Morgan, 2019, 2-3)

De Morgan is here drawing from the classical theory of designation in language to project new intuitions upon logical terms. Yet he does not borrow from such tradition without transforming it. For, given the particular use of contraries he wants to put forward, he will propose an original conception of the referential nature of language, following which a name does not only refer to the things or ideas it represents, but every name refers to every possible idea:

> Every name has a reference to every idea, either affirmative or negative. The term *horse* applies to every thing, either positively or negatively. This (no matter what I am speaking of) either *is* or *is not* a horse. (De Morgan, 2014, 35)

It appears that, by mobilizing complementation of collection of objects in a universe, De Morgan is capable of renewing the traditional theory of designation. However, interpreting terms as linguistic units (i.e., names) is far from ineffective. For in this way, logical terms are seen as *symbols* that can become *the object of algebraic manipulation*, following the principles of the emerging symbolic algebra of his time. De Morgan is fairly explicit on this point when it comes to the algebraic mode of thinking about opposition:

> I hold that the system of formal logic is not well fitted to our mode of using language, until the rules of direct and contrary terms are associated: the words direct and contrary being merely correlative. Those who teach Algebra know how difficult it is to make the student fully aware that a may be the negative quantity, and $-a$ the positive one. There is a want of the similar perception in regard to direct and contrary terms. (De Morgan, 2019, 3)

To sum up, De Morgan's notion of contrariness draws from two sources of originality. On the one hand, an embryonic notion of *class* endowed with an operation of complementation over a particular enclosing class (the universe), providing a stable and unrestricted semantics for contrary names. On the other, a notion of *name* which, although inheriting from classical theories of language, is renewed through a symbolical approach derived from new algebraic practices. In this setting, neither classes nor names have preeminence over each other. While classes promise to provide a solid semantics for names, they are not given directly but only through names. More significantly, the nature of the relations between classes is yet to be defined. For class relations such as unions, intersections or inclusions have no primitive character in De Morgan's early work. In contrast, relations between names are indeed given, precisely *in logical propositions*.

A_1	$X)Y = X.y = y)x$	|	A'	$x)y = x.Y = Y)X$
O_1	$X{:}Y = Xy = y{:}x$	|	O'	$x{:}y = xY = Y{:}X$
E_1	$X.Y = X)y = Y)x$	|	E'	$x.y = x)Y = y)X$
I_1	$XY = X{:}y = Y{:}x$	|	I'	$xy = x{:}Y = y{:}X$

Figure 3: Table of contranominal propositions (De Morgan, 2014, 61)

2.4 ...and back.

After appealing to an interpretation of logical terms simultaneously as names and as classes in definite universes, De Morgan can now propose a reliable notion of contrariness to be added to the other dimensions defining the internal structure of a proposition, namely, quantity, quality, and order. Although sometimes referring to contrary names by prefixing names with the word "not" (as in "not-man"), when it comes to the symbolical notation of contrariness within his logical system, De Morgan chooses to express it by lowercasing name symbols. Thus if 'X' is the symbol for a name like man, not-man will be expressed by its corresponding lowercase: 'x' (cf. De Morgan, 2014, 38).[7]

Extending the initial strategy presented in section 2.2, if contrariness is now applied in every possible way to both terms in each of the six elementary propositions already established, we are left with $2 \times 2 \times 6 = 24$ "apparent modes". Yet, just as the eight possible modes resulting from all the combinations of quantity, quality, and order could be reduced to only six through the equivalence of convertible propositions, De Morgan affirms that these 24 modes can, in turn, be reduced to only eight, through the equivalence of the former three-by-three, as shown in fig. 3.

Significantly, De Morgan gives no proof of this equivalence, neither in his 1846 article "On the structure of the syllogism" (reprinted in De Morgan, 2019) nor in his more elaborated *Formal Logic*, mentioning only that "most readers will readily see the truth of the identities here affirmed" (De Morgan, 2014, 61). However, he proposes the following singular construction instead, as a "mode of illustration" of this fact, to which he moreover attaches a diagrammatical representation of "specimens of the eight standard varieties of assertion" (fig. 4):

> Let U be the name which is the universe of the proposition: and write down in a line as many Us as there are distinct objects to which this name applies. A dozen will do as well for illustration as a million. Under every U which is an X write down X : and x, of course, under all the rest. Follow the same plan with Y. The

[7]De Morgan's notation for contrariness is often criticized as the symptom of his conception of negation's lack of involutive character (cf. for instance Hailperin, 2004, 347). Yet, as one of the leading mathematicians of the nascent abstract algebra, it is implausible that De Morgan has not reflected upon such a decisive point. If he did not choose to symbolize contrariness of names with a standalone symbol, we should conclude instead that he did not want to grant to it the status of an independent operator. We will see that, in De Morgan's view, contrary names, interpreted as class complementarity, do indeed relate to propositional negation; however, we will also see that De Morgan has something else in mind than identifying contrary names with propositional negation as an independent connective or operator based on the semantics of class complementarity over the universe of discourse.

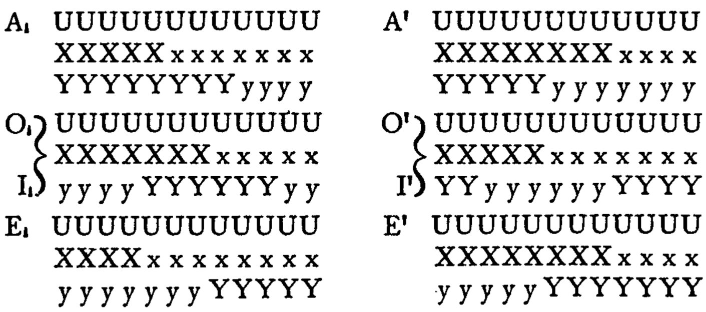

Figure 4: Illustration of "specimens of the eight standard varieties of assertion" (De Morgan, 2014, 61)

> occurrence of letters in the same column shows that they are names of the same object. (De Morgan, 2014, 61)

The absence of a proof should not lead us to conclude that De Morgan's reasoning is not rigorous. For this construction, filling the gap left by the lack of an explicit proof, provides enough information to reconstruct the rather precise objects De Morgan seems to be manipulating in his careful elaboration of a semantics for logical propositions.

How to account, then, for De Morgan's illustration of the mechanisms by which the equivalence between propositional forms can be established? At the risk of incurring anachronism, it can be useful to represent De Morgan's construction in contemporary terms.[8] If we return to the fundamental relation between names and objects presented in the previous section, we can say that to each name n_i of the set of names N corresponds a class C_{n_i} of objects o_j in a universe U. The class of all possible o_j is, however, left entirely undefined, and must not be mistaken for the universe U. Indeed, for De Morgan, U does not comprise all possible o_j, neither is it fixed. As we have seen, for him, the universe is always the universe "of a proposition, or of a name" (De Morgan, 2019, 2). Hence, as a class of objects where other classes could be determined, U is no less the correlate of a name than any other class.[9] The only difference is that, while all the other classes considered are supposed to be (properly)[10] included in a corresponding universe class, De Morgan does not address the thorny issue of the latter's inclusion in an encompassing class.[11] Classes of objects (including

[8]Nothing of what follows should let the reader think that De Morgan is actually manipulating set-theoretical objects as we know them. Set theory is rather the distant consequence than the source of a construction such as De Morgan's. However, this does not mean that his objects are less precise, as manifested by the coherence of his approach and expressive means. It is then this consistency that we intend to restore by translating it into contemporary terms.

[9]See, for instance: "Thus, the universe being mankind, Briton and alien are contraries, as are soldier and civilian, male and female, &c.: the universe being animal, man and brute are contraries, &c." (De Morgan, 2014, 38).

[10]See footnote 18 below.

[11]De Morgan also avoids addressing the possibility for the universe class, or classes in general, to be infinite (cf. De Morgan, 2014, 111). Two implicit requirements seem to govern his conception of classes in this sense: that classes are countable and that complementation over classes is well defined. This certainly explains his sparing use of the symbol u as an empty class, contrary to U: the complement of the universe class does not seem well defined. See, for instance, (De Morgan, 2014, 106, 120), where u, "the contrary of U", is introduced as a symbol

U) result from the fact that one or more objects can be named with the same name. If we represent the naming relation as a pair (n_i, o_j) for a given collection of objects, then, for a name $n_u =$ 'U' chosen as defining the universe, the class $U = C_{\text{'U'}}$ would be determined as the class of o_j under consideration such that $(\text{'U'}, o_j)$ exists.[12]

Once the universe U is thus defined and to some extent assumed to exist, it would be possible to represent—following De Morgan's novel idea that every name refers to every object—all the possible ways of naming all objects in that universe as the Cartesian product $N \times U$. A particular naming configuration over that universe (corresponding, for instance, to a particular language) is then determined by an arbitrary subset R_k of this set.[13] In this setting, a class X determined by the name 'X' $\in N$ can be defined in modern terms as the set $\{o \in U \mid (\text{'X'}, o) \in R_k\}$ for a given R_k. This is consistent with De Morgan's text, where only names, objects, and naming relations are given, and classes are conceived as resulting from those naming relations. More generally, the set R_k, representing a specific collection of naming relations, ensures that to each name n_i in N corresponds a class of objects o_j in U (cf. fig. 5). But then, given the arbitrary character of the set of names N, nothing prevents us from introducing a name for any class of objects in U. In particular, if you allow yourself to resort to class complementation over U, then there are no obstacles to introducing contrary names in a systematic way, which is precisely what De Morgan proposes to do.

The connection between names and classes is thus governed by the abstract naming relation R. But a given R_k does more than just map names and classes. It also determines *a relation between different names* and is, as such, the source of *propositions*. For if R happens to include, say, both $(\text{'X'}, o_1)$ and $(\text{'Y'}, o_1)$, then it is possible to affirm that 'X is Y', following De Morgan's interpretation of logical propositions as 'this one X (i.e., o_1) is this one Y (i.e., o_1)'. Since

by which "we can only denote non-existence", with no clear class semantics. In practice, De Morgan usually refers to finite classes, except maybe in his treatment of probabilities (cf. De Morgan, 2014, 213-214).

[12]Given the undefined character of the class of all objects, the existence of the class U is somewhat assumed by De Morgan.

[13]De Morgan himself insists on the arbitrary character of naming relations. For instance: "A class of objects has a sub-class contained within it, the individuals of which are distinguished from all others of the class by something common to them and them only. [. . .] it will more often happen that a distinctive characteristic, belonging to some only, gives no distinctive name to those *some*, which still remain an unnamed *some* out of the whole, to be separated by the description of their characteristic when wanted, instead of being the *all* of a name invented to express them, and them alone of their class." (De Morgan, 2014, 39).

This point becomes even stronger when it comes to contrary names, where that arbitrariness is associated with particular languages: "Whether a language will happen to possess the name B, or *b*, or both, depends on circumstances of which logical preference is never one, except in treatises of science. The English may possess a term for B, the French only for *b*: so that the same idea must be presented in an affirmative form to an Englishman, as in 'every A is B,' and in a negative one to a Frenchman, as 'no A is *b*.' " (De Morgan, 2014, 40). Interestingly, this fact constitutes a source of critique of the syllogistic treatment of logic, for "it is an accident of language whether a proposition is universal or particular, positive or negative." (De Morgan, 2014, 40). We can easily imagine how De Morgan's association between names and classes aims to overcome this pitfall.

However, in our reconstruction, the set R is not entirely arbitrary, since for a given name 'X' and object o_j the pairs $(\text{'X'}, o_j)$ and $(\text{'x'}, o_j)$ cannot both be in R. This departure from the pure arbitrary character of language comes from the fact that contrary names have their source on class complementation, which De Morgan introduces as a principle foreign to language, but also the source of logical properties.

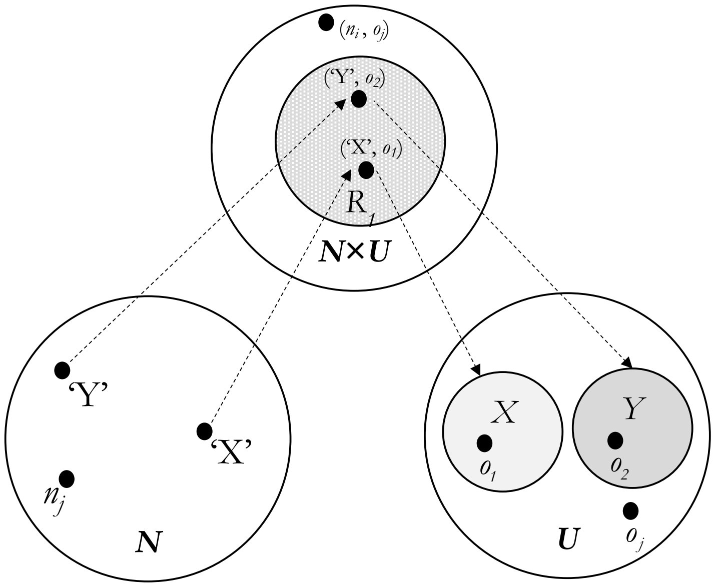

Figure 5: Illustration of a set set-theoretical reconstruction of De Morgan's relation between names and classes.

the relation between names or terms is precisely what constitutes a proposition, we propose to call such a relation *propositional relation*, to distinguish it from *propositional forms* or modes. The latter refer to traditional syllogistic propositions (A, E, I, O), involving quantification in a way that comprises multiple propositional relations (in the sense just proposed). A given abstract relation R_k is not to be confused with any of the two. In particular, R_k is a collection of pairs (n_i, o_j), while a propositional relation, in our terms, is given by a relation between two of those pairs belonging to R_k and sharing the same o_j.[14]

Certainly, by construction, propositional relations determined by R_k, traditionally expressed through logical propositions, can find in the current setting an "objective" correlate through classes of U. However, propositions do not correspond to classes (as we would be tempted to think from a contemporary perspective), but are rather associated to relations between classes: to every R_k corresponds a particular configuration of classes in U. And significantly, there are no traces in De Morgan's early work that class relations other than complementation (such as inclusion, union or intersection), are treated as primitives. On the contrary, the relation between names will be invariably dealt with through the lens of propositions by a sophisticated reworking of their syllogistic understanding.

What is important for now is that this contemporary picture provides a fairly

[14]This conception of propositions as relations (and more precisely, of the propositional copula *is* as governed by an abstract relation) will occupy a central place in De Morgan's later developments (cf. Merrill, 1990; Sánchez Valencia, 2004)). In this sense, the centrality of relations in our reconstruction remains faithful to De Morgan's approach. However, with this, we do not mean to attribute to De Morgan a contemporary understanding of relations (as subsets of a Cartesian product) or of propositions (as identified with those or other related subsets).

faithful translation of the objects presented in the illustration diagrammatized in fig. 4, which govern De Morgan's reasoning. Indeed, each one of the six "specimens" De Morgan presents here corresponds to a different R_k, in this case, for a given $N = \{\text{'U'}, \text{'X'}, \text{'Y'}\}$ and a $U = \{o_j \mid 1 \leq j \leq 12\}$. Each of the "columns" to which he refers in his description features all the names associated with each one of the objects o_j within each of the six "specimens", with the name 'U' being attributed by definition to all the objects in the universe. In other words, each column corresponds to the set of names $\{n \in N \mid (n, o_j)\}$ for a different o_j. Notice how, in De Morgan's description, the contrary names 'x' and 'y' are added in a systematic way only at a later stage, as reflected by our reconstruction.

We are now almost ready to resume our initial question: how can the equivalence three-by-three between the 24 propositional modes be established? 'Almost', because while the place of propositional relations seems clearly determined, *propositional forms* or modes do not yet find a representation in our setting. For the moment, the propositional relations are determined through more or less arbitrary abstract relations R_k. But, following De Morgan's illustration, these are not proper propositional forms, such as A, E, I or O, but only "specimens" of them. Indeed, if for instance in the first of the six cases featured in fig. 3, we replace the first column $\{\text{'U'}, \text{'X'}, \text{'Y'}\}$ with $\{\text{'U'}, \text{'x'}, \text{'Y'}\}$, we would still be dealing with a universal affirmative proposition of the form A: Every X is Y. As De Morgan says, with respect to such specimens, propositional forms should be understood as "standard varieties of assertion" (De Morgan, 2014, 61). In other words, as specific collections of such specimens.

In our reconstruction, the representation of such collections requires that we consider the set where all the R_k live, namely the power set of the Cartesian product of N and U: $\mathcal{P}(N \times U)$. As collections of such elements, De Morgan's 24 apparent modes can now be represented as *subsets of this power set*. It follows that, under this representation, two propositional forms are equivalent if they correspond to the same class of $R_k \in \mathcal{P}(N \times U)$. Thus, class identity in $\mathcal{P}(N \times U)$ provides a natural semantics for the equivalence of propositional forms introduced by De Morgan. In particular, such a semantics allows us to make explicit that if we have the right to consider, for instance, X)Y, X.y and x)y as one and the same thing and this equivalence fully characterize a propositional form, for De Morgan, such an entity is of a different kind than classes such as X or Y, and than propositional relations connecting those classes.

2.5 The Triple Root of Logical Negation

It is essential to insist that, with this reconstruction, we do not mean to attribute to De Morgan a contemporary understanding of logic, where propositions would be reducible to sets, let alone the very idea of modern semantics, altogether foreign to his work. If anything, our attempt shows, as announced, that a contemporary reconstruction intending to remain faithful to De Morgan's original system is far from simple. In particular, unlike our current understanding of propositional logic, at their most basic level, classes do not correspond to propositions but names as sub-propositional entities. Accounting for propositions in De Morgan's terms requires to consider a much more complex structure instead, made of successive layers: on top of that nominal plinth that is the universe U defining the first layer, we find a second layer, where propositional relations live, determined by somewhat arbitrary classes within $N \times U$; and finally, a third one, where propositional forms can be represented as classes of such classes, identifiable within $\mathcal{P}(N \times U)$.

It is clear that neither De Morgan had access to set-theoretical tools such as the Cartesian product or the power set, nor can we find any trace of such operations in his work. Notwithstanding, speaking about classes for these three different layers does not seem totally unreasonable, for they are all characterized by three different kinds of objects which De Morgan takes particular care not to confuse and which correspond rather faithfully to the ones characterized through our sets: names/classes of objects, relations, collections of relations. But more significantly, at each one of those layers, De Morgan resorts to a different form of negation, all of which, just like contrariness for names, correspond to the behavior of class complementation.[15] In the case of contrariness, the reference to complementarity was fairly explicit. In the cases of propositional relations and propositional forms, the situation is more subtle. Let us start by taking a look at the former.

Once the equivalence three-by-three of the 24 apparent modes is established, it is possible to identify a symmetry within what De Morgan calls the "enlarged view" of propositions (De Morgan, 2014, 60), that is, the extended collection of modes resulting from the consideration of nominal contrariness modulo propositional equivalence. Such a symmetry is already suggested by De Morgan's presentation of those eight forms in two opposite columns (fig. 3). In De Morgan's words, the reason behind such a presentation is that "These eight modes may all be derived from the four Aristotelian modes by changing both terms into contraries" (De Morgan, 2019, 5). Indeed, if we take any of the twelve 'apparent modes' corresponding to the four Aristotelian modes arranged in the first column, we can obtain the corresponding mode in the opposite column just by changing terms into contraries.

Although from the perspective of contrariness, this symmetrical relation appears as the result of a compound operation (simultaneous change of subject and predicate names into their contraries), it seems clear that De Morgan attached a great significance to it as one whole, since he gave a particular name and notation to it. Thus, the modes obtained by changing both terms into their contrary are called *contranominals*. More significantly, owing to an analogy with the notation of contrary names, in his 1846 article, De Morgan proposed to denote them with corresponding lowercase letters (e.g., 'a' for the contranominal of 'A') (De Morgan, 2019, 5). This notation suggests that, with contranominality, De Morgan is raising to the level of propositions the contrariness introduced at the level of names. And indeed, in our reconstruction, contranominality can very well be represented by a simple complementation, namely the complement of R_k over the set $N \times U$.

The fact that, later on, in his *Formal Logic*, De Morgan decided to substitute the lowercase notation for contranominals by the use of diacritics is undoubtedly a sign that, despite the same underlying mechanism (class complementation), both operations should not be confused due to the different nature of their objects. Thus, after affecting the old symbols A, E, I, O of inverted accents to obtain A₍, E₍, I₍, O₍, he notates their respective contranominals with regular

[15] As we will see, De Morgan will also interpret inferential properties of propositions through inclusion over these classes. See section § 3.1 below.

accents, pointing in this way to the underlying symmetry: A', E', I', O' (see fig. 3).

From the point of view of dual mechanisms in De Morgan's logical framework, the fact that contranominality may be interpreted as a form of complementation (over a different class than the Universe) is highly significant. For, in contemporary terms, *contranominality behaves with respect to contrariness as an external to an internal negation*, motivating the emergence of dual mechanisms at the core of his system. Indeed, as fig. 3 clearly shows, the interaction between contranominality and contrariness is such that, if considered as operators, the four syllogistic relations appear as duals of themselves.

However, despite their original symbolic expression, traditional syllogistic relations between propositional forms are devoid of operational character in De Morgan's system, no less than contranominality itself, which, as such, is not expressed by any standalone symbol. Normal and inverted accents are ways to identify (dual) objects rather than denote operations to be performed on an object to obtain its dual.[16] Yet, if only as a purely implicit operation behind the symmetry relating contranominal propositions, and suggested by analogy to the well-established contrariness, contranominality appears as another form of logical negation present in De Morgan's system. Interestingly, the fact that contrariness and contranominality do not coincide due to the difference in the nature of their objects is a desired feature of De Morgan's approach, which we have already witnessed in his treatment of order (§ 2.2). Just as in that case, we see that an interaction is staged between propositional and sub-propositional units, from which logical properties are expected to emerge. Significantly, the emergent properties suggested by the interaction between the two forms of negation take here the form of *dual* properties.

However, as we have announced, contrariness and contranominality are not the only forms of negation identifiable in De Morgan's edifice. A third form, associated with the level of propositional forms, can still be recognized. Yet this requires that we address the question of conjunction and disjunction in his system.

3 Conjunction and Disjunction

Stemming from the Stoic tradition, the treatment of conjunction and disjunction as related principles for connecting propositions occupied a relatively marginal place in the Aristotelian tradition of syllogistics, where conditionals have been invariably prioritized (cf. for instance, Bonevac and Dever, 2012). Standing on syllogistic bases, it is not surprising then that a clear, systematic, and joint assessment of conjunction and disjunction is nowhere to be found in De Morgan's early logical work. More surprising can be the fact that neither can we find an introduction of those principles through typical class operations, such as intersection and union, as we would retrospectively expect from a work that

[16] Operators are nonetheless not entirely absent in this setting. Indeed, De Morgan introduces explicit operators for contrariness of the subject (S) and the predicate (P), as well as order (T) and quality (F), and even an identity operator (L), all of which act upon propositions. His aim is to show, through algebraic manipulations over such operators, that contrariness (S and P) and contranominality (SP) are enough to operate any possible change from a propositional form to another (cf. De Morgan, 2014, 63-65).

can be credited for introducing a class semantics for logic. However, as already mentioned, class operations like intersection and union also lack any explicit introduction or systematic manipulation in this work. And yet, rudimentary as it may be, De Morgan's incipient class semantics makes disjunctive and conjunctive aspects omnipresent throughout his new logical construction, motivating highly original views, including the dual properties directly associated with them.

An early example of this situation can be found at the very first stages of his construction in *Formal Logic*, where De Morgan resorts to his nominal interpretation of the copula in the framework of an argument about convertible propositions. De Morgan affirms:

> If 'this X be a Y' it is one Y only: it is 'this X is *either* the first Y, *or* the second Y, *or* the third Y, &c.' [...] But if it be 'this X is not a Y,' we have 'this X is not the first Y, *and* it is not the second Y, *and* it is not the third Y, &c.' The affirmation is what is commonly called *disjunctive*, the negation *conjunctive*. A disjunctive negation would be no proposition at all, except that one and the same thing cannot be two different things: any X is either not the first Y *or* not the second Y. And in like manner a conjunctive affirmation would be an impossibility: it would state that one thing is two or more different things. (De Morgan, 2014, 59)

Regardless of the context in which it appears and the relatively marginal argument De Morgan wants to advance with it, this remark already exhibits a few remarkable features concerning the place of conjunction and disjunction in his system. First, conjunctive and disjunctive properties result from the action of negation in a setting where logical terms are given an extensional interpretation as classes of objects. Moreover, that action induces a dual behavior on those properties. Finally, conjunction and disjunction are not here conceived as primitive connectives explicitly introduced and manipulated, but rather as implicit properties of that extensional setting, related in some way to quantification conditions that also exhibit a dual behavior with respect to negation.[17]

Yet the conjunction and disjunction here at play do not concern propositions, not even names. Instead, they characterize relations between individuals. As such, one can not expect them to be easily lifted to the propositional level in De Morgan's stratified system. However, a propositional case of conjunctive and disjunctive properties exhibiting analogous dual behaviors can indeed be found in *Formal Logic*, which, while in principle unrelated to the one just seen, seems to result from yet another use of class complementation.

[17] De Morgan will state this idea more clearly at the opening of his 1850 paper, where the strict correspondance between quantification and conjunctive and disjunctive properties is made explicit: "The distinction of universal and particular is that of conjunctive and disjunctive; the universal speaks conjunctively, the particular disjunctively, of the same set." (De Morgan, 2019, 25). De Morgan's argument strongly recalls Peirce's later pioneering treatment of quantification as logical sums and products (cf. Beatty, 1969). Incidentally, the unusual idea of considering conjunction and disjunction as internal properties of negation and affirmation, and existential and universal quantification, respectively, strongly resonates with the notions of expansivity and recessivity and their relation to polarity put forward by J. Y. Girard in contemporary logic (cf. Girard, 2011, §§ 2.1.4, 2.A.4).

3.1 Complex Propositions

If we return to De Morgan's progressive construction presented in the last section, we can see that, even if already extending the traditional syllogistic paradigm, the eight elementary propositional forms of the "enlarged view" are not the only modes De Morgan can conceive within his new logical setting. They are only the *simple* forms. However, already in the first page of the Preface of his *Formal Logic*, De Morgan announced the introduction of *complex* propositions, not without already warning against any possible priority of the former over the latter, other than purely historical. In particular, complex propositions *are not just the free composition of simple ones.* Their introduction is motivated by a different problem.

As in the classic square of oppositions, the affirmation of propositions of the enlarged view entails the affirmation and denial of other propositions. This circumstance is, by the way, responsible for much of the inferential capacities of the system of syllogistic propositions. Take, for instance, the propositional form $\mathrm{A}_{\prime}$ ('Every X is Y'): its affirmation implies the denial of $\mathrm{O}_{\prime}$ ('Some X is not Y'), $\mathrm{E}_{\prime}$ ('No X is Y') and E' ('No x is y'), as well as the affirmation of $\mathrm{I}_{\prime}$ ('Some X is Y') and I' ('Some x is y').[18] However, unlike the traditional syllogistic setting, once a proposition is affirmed in the enlarged view, *it is always possible to find propositions indifferent to that affirmation.* For instance, in our example, by affirming $\mathrm{A}_{\prime}$, nothing is said about A' or O': it could very well be that every Y is X or that some Y is not X. In other terms, the extension to contranominals reveals that simple propositions are underdetermined or, in the words of De Morgan, *incomplete* (De Morgan, 2014, 56).

To deal with this ambiguity, it suffices, however, to affirm the initial proposition and one of the alternatives or "concomitants", as De Morgan calls them. In our example, by affirming both $\mathrm{A}_{\prime}$ and A' (or, alternatively, $\mathrm{A}_{\prime}$ and O'), the indeterminacy is removed, and all eight simple forms are now either affirmed or denied.

A free composition of the eight simple propositional forms would yield 256 complex forms. However, as we have mentioned, complex forms are not introduced by De Morgan as a way to grasp compositionality, but concomitance, as defined above. Hence, by considering all the cases of concomitance within simple propositions (as shown in the table of fig. 6), De Morgan concludes they can be captured by only seven complex forms, determined by the coexistence of simple ones.

The first of those seven forms corresponds to a marginal case from the logical viewpoint in which none of the universal propositions are true, and hence all particulars are.[19] De Morgan will designate this form with a P (for 'particular'), symbolizing its content as follows: $\mathrm{O}'+\mathrm{O}_{\prime}+\mathrm{I}'+\mathrm{I}_{\prime}$, "denoting coexistence of simple

[18] Interestingly, the fact that De Morgan considers I' affirmed when $\mathrm{A}_{\prime}$ is affirmed reveals implicit assumptions of his system, namely that neither the names under consideration (i.e., 'X' and 'Y'), nor their respective contraries define the same class as the Universe, but only proper subclasses. For if X, for instance, would be equal to U, and $\mathrm{A}_{\prime}$ affirmed, then I' would be affirmed without there being any 'x'. The affirmation of I' would thus not have existential import, which De Morgan explicitly rejects: "The existence of the terms must be first settled, and then the truth or falsehood of the proposition. The affirmative proposition requires the existence of both terms" (De Morgan, 2014, 111). This will have consequences in our contemporary reconstruction (see p. 25).

[19] As De Morgan points out, this case bears little importance within the syllogistic tradition and will not play a significant role in his own perspective (De Morgan, 2014, 66).

	Denies	Contains	Is indifferent to		Denies	Is contained in	Is indifferent to
$A_{\prime}$	$O_{\prime}E_{\prime}E'$	$I_{\prime}I'$	$A'O'$	$O_{\prime}$	$A_{\prime}$	$E_{\prime}E'$	$A'O'I_{\prime}I'$
A'	$O'E'E_{\prime}$	$I'I_{\prime}$	$A_{\prime}O_{\prime}$	O'	A'	$E'E_{\prime}$	$A_{\prime}O_{\prime}I'I_{\prime}$
$E_{\prime}$	$I_{\prime}A_{\prime}A'$	$O_{\prime}O'$	$E'I'$	$I_{\prime}$	$E_{\prime}$	$A_{\prime}A'$	$E'I'O_{\prime}O'$
E'	$I'A'A_{\prime}$	$O'O_{\prime}$	$E_{\prime}I_{\prime}$	I'	E'	$A'A_{\prime}$	$E_{\prime}I_{\prime}O'O_{\prime}$

Figure 6: Table of concomitant propositions (De Morgan, 2014, 63)

propositions by writing + between their several letters" (De Morgan, 2014, 66). In the six remaining cases, a universal coexist with one of its concomitants. De Morgan represents them as follows: $A_{\prime} + A'$; $A_{\prime} + O'$; $A' + O_{\prime}$; $E_{\prime} + E'$; $E_{\prime} + I'$; $E' + I_{\prime}$, designating them with the respective notations: D, $D_{\prime}$, D', C, $C_{\prime}$ and C'.

If the meaning of De Morgan's introduction of complex propositions does not become immediately apparent, we can pay attention to their interpretation in terms of relations between the classes X and Y in the Universe. It will then appear that, unlike simple propositions, De Morgan's *complex propositions represent meaningful and non-ambiguous relations between the classes corresponding to the terms of the propositions*. Thus, while P refers to the case in which the classes X and Y partially intersect, without covering the Universe, the proposition D ($= A_{\prime} + A'$) represents the identity of X and Y, and C ($= E_{\prime} + E'$) represents the relation of contrariness between X and Y, conceived as the identity of one and the contrary of the other (X and Y thus partitioning the Universe). Following the same interpretation, $D_{\prime}$ and D' represent the strict inclusion of X in Y and of Y in X, respectively, while $C_{\prime}$ represents the strict inclusion of a term in the contrary of the other and C' that of the contrary of one term in the other. All of which De Morgan expresses by naming the propositions D, $D_{\prime}$ and D' *identical*, *subidentical* and *superidentical* respectively, while calling *contrary*, *subcontrary* and *supercontrary* the propositions C, $C_{\prime}$ and C'.

It thus appears that, with complex propositions, De Morgan finds the way of representing at the propositional level a significant number of relations between classes corresponding to the semantics to which he has attached the old syllogistic setting, namely *complementation*, *inclusion*, and *identity*. This means that, even if those relations might have more or less silently guided De Morgan through his construction of a new logical framework, such construction is not formally based on class operations. Rather the contrary seems to be the case: it is his construction of a propositional logic, resulting from successive extensions based on a notion of opposition of algebraic origin, that encourages a formal theory of classes to take shape.

Significantly, De Morgan attributes the priority for the introduction of complex propositions to the French mathematician Joseph Gergonne in his *Essaie de dialectique rationnelle* Gergonne (1816-1817), even if he affirms having read Gergonne's treatise only after his own memoir was published in the *Transactions of the Cambridge Society*.[20] As it is known, the French mathematician had

[20]For De Morgan, in Gergonne's paper "There is the idea, and some formal use, of a complex proposition[...] M. Gergonne's complex propositions, such as they are, are used in a manner resembling that in chapter V, of this work, though requiring a separate *tâtonnement* for many things the analogues of which appear as connected results of my system. Accordingly, I am

made significant contributions to the theory of duality in projective geometry, starting with a series of papers in 1810. In his 1816 logical treatise, and inspired by Euler's diagrams, Gergonne proposes an extensional understanding of "ideas" or "notions", immediately interpreted in geometrical terms through the intersection, inclusion, or exclusion of figures on a plane (and of circles in particular). He then introduced a direct notation for several possible cases, namely H for the case where two notions (or the circles representing them) do not intersect, X for partial intersection, I for identity, C for inclusion of the subject in the predicate and Ↄ for inclusion of the predicate in the subject. Armed with these relations,[21] Gergonne was able to fully characterize the conditions for the truths of traditional syllogistic propositions.

However, as intimately connected with syllogistic propositions as his five relations may be, Gergonne never envisaged the possibility of attributing propositional status to them. Rather the opposite, since he actually pointed out the incapacity of existing languages to directly represent the relations presented by those elementary cases:

> Il n'est aucune langue connue dans laquelle une proposition exprime précisément et exclusivement dans lequel de nos cinq cas se trouvent les deux termes, qui la composent; une telle langue, si elle existait, serait bien plus précise que les nôtres; elle-aurait cinq sortes de propositions; et sa dialectique serait toute différente de celle de nos langues. (Gergonne, 1816-1817, 199)

By representing those class configurations as fully-fledged propositions, De Morgan actually hints at, if not creates, that very language. Certainly, complex propositions cannot represent those configurations directly, but only through the composition of several other ("simple") propositions. However, as he constantly recalls, simple propositions have no privilege in this respect. For if we consider complex propositions as the choice for a simple proposition of one of its concomitants, *simple propositions can be thought of, in turn, as defined by the alternative between complex ones.* Take, for instance, $\mathrm{D}_{\prime}$, which is defined as the coexistence of $\mathrm{A}_{\prime}$ and O', and D as that of $\mathrm{A}_{\prime}$ and A'. O' and A' being the only concomitants of $\mathrm{A}_{\prime}$, it follows that the latter can be sufficiently characterized by the alternative between $\mathrm{D}_{\prime}$ and D.

In this way, De Morgan proposes a novel conception of logical propositions in which no proposition is, strictly speaking, simple, being all composed of several others. And yet, while not simple, De Morgan's setting has the virtue of conceiving the compound nature of propositions as elementary. In other words, composition is less seen as an external relation between propositions than an internal constitutive principle of elementary logical propositions as such. We may speak about *articulation* to distinguish composition as an internal principle of propositions ($\mathrm{D}_{\prime}$ being composed *of* $\mathrm{A}_{\prime}$ and O') from the usual notion of composition as the external combination of two propositions ($\mathrm{A}_{\prime}$ being composed *with* O'). We could then say that, even if elementary, for De Morgan, every proposition is articulated.

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bound to attribute to M. Gergonne the first publication of the idea of *a* complex syllogism, and of the comparison of the simple one with it." (De Morgan, 2014, 324).

[21] Which would much later become known as the "Gergonne relations". See Faris (1955); Grattan-Guinness (1977).

3.2 Dual Behavior of Propositional Disjunction and Conjunction

Armed with this new logical apparatus, De Morgan is capable of providing an original and simplified account of syllogistic inference, governed by the interaction between class relations (identity, strict inclusion, and complementation) at the nominal level. Indeed, at the opening of the first of the two chapters on the syllogism in his *Formal Logic*, De Morgan distinguishes the simple and complex syllogisms, the latter being "one in which two complex propositions produce the affirmation or denial of a third complex proposition" (De Morgan, 2014, 76). For instance, from two premisses, both of form Dı, that is, X subidentical of (i.e., a class properly included in the class) Y, and Y subidentical of Z, it is possible to conclude that X is subidentical of Z, that is: Dı. De Morgan expresses this situation with the sequence of symbols: DıDıDı. After exploring all the combinations for valid syllogisms, he can present symmetric relations between the syllogism themselves, by changing all their propositions into their respective contranominals. Thus, if CıD'Cı is a valid syllogism, so is C'DıC', and conversely (De Morgan, 2014, 78-79).

It becomes thus clear that, after De Morgan's sophisticated reworking of the propositional setting, syllogistic inference can receive a new interpretation based on the transitivity of class inclusion in the Universe of discourse, also taking advantage of the involutive behavior of contranominality. In return, class relations such as inclusion can now be represented logically at the propositional level. Not only will De Morgan add this interpretation to the traditional understanding of syllogistic inference of simple propositions, but he will also argue for the preeminence of complex over simple propositions in terms of strength, clearness, easiness, evidence and analytical power.

An account of De Morgan's reconstruction of classical syllogistic inference falls outside the scope of the present paper.[22] However, the interplay between simple and complex propositions involves a profound insight into conjunction and disjunction as dual logical principles. For, when put side by side, both kinds of propositions reveal a new symmetry in the logical space. While both are conceived as articulated, their articulation is not exactly of the same nature: as we have seen, complex propositions are defined as the *coexistence* of simple ones, simple through a *concomitance* motivating an alternative between complex ones. Interestingly, those two articulation principles present *an inherently dual behavior*, from which De Morgan will draw the source of a proper logical duality between propositions.

The first step in that direction is given by De Morgan's interpretation of those two articulation principles precisely in terms of *conjunction* and *disjunction*:

> But it will be said, surely the complex proposition requires the conjunctive existence of two simple ones: Dı=Aı+ O'; and is therefore compound at least. I answer that, on the other hand, the simple proposition requires the disjunctive existence of two complex ones: as Aı= Dı or D. (De Morgan, 2014, 85)

It is important to insist that conjunction and disjunction are thus introduced, not primarily as ways of freely combining propositions but as correlative forms

[22] For a comprehensive treatment of De Morgan's view on syllogistic inference, one could consult, for instance, Merrill (1990); Sánchez Valencia (1997, 2004).

of *defining* them. As in the case of the conjunctive nature of negation and disjunctive affirmation mentioned above, complex propositions are inherently conjunctive. Not, as we would tend to think, because they can be composed with other complex propositions of the same nature through a binary operator obeying the rules of conjunction, but because their very definition is to be the conjunction of two simple ones (e.g., $\mathrm{D}_{\shortmid} = \mathrm{A}_{\shortmid}$ *and* $\mathrm{O}^{\shortmid}$). Correlatively, simple propositions are intrinsically disjunctive in that, considered as the other side of complex ones, as it were, they appear as articulated through disjunctive relations between them (e.g., $\mathrm{A}_{\shortmid} = \mathrm{D}_{\shortmid}$ *or* D), and therefore can be properly defined in this way. Conjunction and disjunction are so intimately related to these correlative principles for defining propositions that De Morgan proposes to abandon the provisory names "simple" and "complex"—misleadingly focusing on the external composition of simple ones—by "disjunctive" and "conjunctive":

> [...] I think it may be allowed to treat the words simple and complex as only of historical reference, and to consider the first as disjunctively connected with the second, the second as conjunctively connected with the first, in the manner above noted. [...] If the plan which I propose should gain any reception, I should imagine that *disjunctive* and *conjunctive* would be the names given to the classes which I have called simple and complex: the conjunctive composed of several of the disjunctive, the disjunctive consisting of one or the other out of several of the conjunctive. (De Morgan, 2014, 86)

In De Morgan's setting, conjunction and disjunction emerge, then, as the defining property of two correlative classes of propositions which are like inverted perspectives into a common semantics. Two sides of the same coin, related by the fact that two groups of elementary propositions are defined in terms of one another. Significantly, in this passage, De Morgan also refers to classes to speak, not of the classes of objects in the Universe, but of propositions themselves ("the classes which I have called simple and complex"). We will have the opportunity to return to the possible sense of this expression. What is important for now is that, from this correlative definition of conjunctive and disjunctive propositions to establishing a properly propositional duality, it is but one short step. All that is needed is to identify yet another principle that systematically converts one into another.

Such a principle is suggested by a specific use of the notion of *denial* applied to complex propositions. Thus, if $\mathrm{D}_{\shortmid}$ "affirms" $\mathrm{A}_{\shortmid}$ *and* $\mathrm{O}^{\shortmid}$[23], De Morgan writes that the "denial of" $\mathrm{D}_{\shortmid}$ affirms $\mathrm{A}^{\shortmid}$ *or* $\mathrm{O}_{\shortmid}$. Accordingly, the denial of all complex propositions results in the exchange of *and* and *or*, together with the substitution of the corresponding simple propositions by their respective contranominals, except for D and C, where the change of universals into particulars is also required (cf. left column of fig. 7).

De Morgan's approach seems here undeniably guided by mechanisms "akin to duality". Indeed, denying complex propositions introduces a novel idea in his propositional framework since, defined as the articulation of two other (simple) propositions, such a denial does not concern each of these individually, but their very (conjunctive) articulation. The exchange it entails from conjunctive

[23] Where "affirms" takes the place of the sign = De Morgan used in defining $\mathrm{D}_{\shortmid}$.

D_{ι} affirms A_{ι} *and* O'	A_{ι} affirms D_{ι} *or* D
D — A_{ι} and A'	A — D_{ι} or D or D'
D' — A' and O_{ι}	A' — D' or D
C_{ι} — E_{ι} and I'	E_{ι} — C_{ι} or C
C — E_{ι} and E'	E — C_{ι} or C or C'
C' — E' and I_{ι}	E' — C' or C
Denial of D_{ι} — A' *or* O_{ι}	O_{ι} denies D_{ι} *and* D
— D — O' or O_{ι}	O — D_{ι} and D, or D' and D
— D' — A_{ι} or O'	O' — D' and D
— C_{ι} — E' or I_{ι}	I_{ι} — C_{ι} and C
— C — I' or I_{ι}	I — C_{ι} and C or C' and C
— C' — E_{ι} or I'	I' — C' and C

Figure 7: Table of dual propositions (De Morgan, 2014, 69)

to disjunctive articulation behaves then as an external negation with respect to contranominality, inducing a duality between the two articulation principles.

De Morgan considered a symmetric extension of these properties to the case of simple propositions, presented side by side in a double-column table (fig. 7). Such a table exhibits a general tendency, and even a strong desire, for a global systematicity in elaborating a new propositional setting for logic, guided by principles and mechanisms obeying dual behaviors. However, the table also manifests the multiple limits De Morgan experienced to attain that goal. Indeed, we have already mentioned the need to exchange universals for particulars in some cases, which suggests that internal negation in the case of complex propositions is not reducible to contranominality. This is further supported by the fact that, on the side of simple propositions, the corresponding internal negation does not involve either of them, but what De Morgan simply calls denial. Also, the role of external negation in simple propositions is not played by denial but by a particularization of universals.

Those are not the only incongruities revealed by fig. 7. However, the cumbrous aspect of this attempt should not hide a possible underlying systematicity of De Morgan's objects and thought. In the last part of this section we propose to restore some aspects of that systematic character, in line with the contemporary reconstruction proposed in the previous section.

3.3 Propositions as Classes

The meaning of propositional conjunction and disjunction is ultimately related to the properties of the classes of individuals in the Universe underlying them (e.g., the conjunction of A_{ι} and O' defining D_{ι} represents the identity of classes X and Y in the Universe of the proposition U). Nevertheless, that meaning is neither reducible to elementary operations on *those* classes nor defined or conceived through truth-functional operators between two individuals or atomic entities. Accordingly, the duality here at stake cannot be directly attributed to the relations between classes of objects in the Universe. Indeed, propositional conjunction and disjunction do not correspond to the intersection and the union of the classes X and Y, no more than the principle converting one into the other corresponds to the complement of those classes over the Universe. And yet, as we have seen, De Morgan refers to simple and complex propositions as classes themselves. It is then unlikely that their dual behavior is entirely unrelated to a class semantics, as elusive or imprecise as the latter may be.

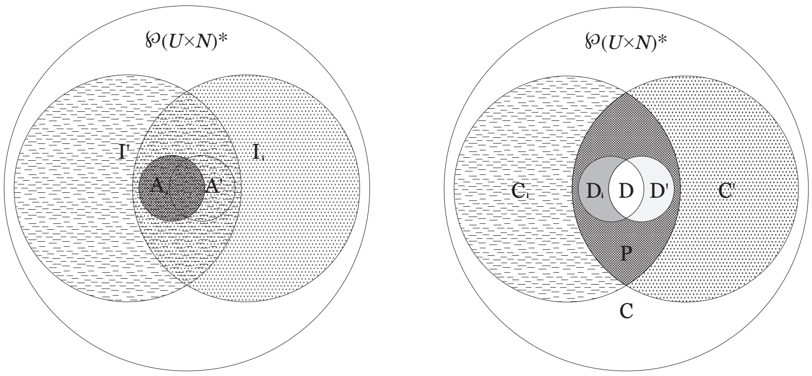
We have suggested that propositional forms or modes, for De Morgan, as "standard varieties of assertion", could be interpreted as classes of relations $R \in \mathcal{P}(N \times U)$. It appears now that the dual properties De Morgan presents between complex and simple propositions can indeed be understood as class relations, but of classes *within this particular space*. Or more precisely, within a normalized version $\mathcal{P}(N \times U)^*$ of such a space, excluding degenerate cases of R for which either of the classes X, Y, or their contraries, would be empty[24].

Take, for instance, the eight disjunctive of simple propositions of the "enlarged view". If we define such propositional forms as subclasses of $\mathcal{P}(N \times U)^*$ in such a way that they capture the collections of "specimens" by which De Morgan illustrates them in fig. 4, then the eight resulting classes will cover that space. Interestingly, the relations between those classes can formally capture the most significant properties of De Morgan's system. Starting with the entailment between different forms, like Aˌ entailing (or "containing") Iˌ (corresponding to subalternation in the classical square of oppositions), which can be interpreted as the inclusion of Aˌ in Iˌ, such that if a given R belongs to the former, it also belongs to the latter. More significantly, the cases of concomitance motivating De Morgan's introduction of conjunctive propositions correspond here to intersections between the classes of that cover.

Due to its multiple intersections, the cover of $\mathcal{P}(N \times U)^*$ produced by disjunctive propositions does not constitute a partition. In contrast, defined precisely out of those intersections, conjunctive or complex propositions do perform a proper partition of this space (cf. fig. 8), thus avoiding ambiguity and indeterminacy. De Morgan's construction corresponds so closely to the partitioning of this space that it is unlikely that he has not been implicitly guided by some version of this property. Whatever the case may be, the relations of intersection and union between all these classes (conjunctive and disjunctive) provide an adequate semantics for propositional conjunction and disjunction as they emerge in De Morgan's logic.

With this interpretation in mind, it is easy to identify the principle exchanging conjunction and disjunction, which is no other than *complementation over* $\mathcal{P}(N \times U)^*$, exchanging intersection and union, and producing the expected dual effects, including those related to the inclusion between classes, corresponding to the inferential properties already mentioned. Significantly, such complementation perfectly corresponds to De Morgan's notion of *contradictory denial*. That is, the relation between propositional forms inherited from traditional syllogistics following which "affirmation of one is denial of the other, and denial of one is affirmation of the other" (De Morgan, 2014, 5). This contradictory version of denial—the dual behavior of which with respect to quantity and quality in the syllogistic framework we have already pointed out—can indeed restore a relatively simple dual structure underlying De Morgan's system of simple and complex propositions (fig. 7). To this end, all we need is, first to interpret the denial of complex propositions in terms of this contradictory denial, and second, consider Oˌ as the contradictory denial of Aˌ, Oˈ as that of Aˈ, Iˌ that of Eˌ, and

[24]See footnote 18.



(a) Simple Propositions (b) Complex Propositions

Figure 8: Different covers of the propositional space $\mathcal{P}(N \times U)^*$. In (a), E and O are to be seen as the complements of I and A, respectively, maintaining the corresponding accents (e.g., O' is the complement of A'). In (b), each proposition covers a non-overlapping region, performing a partition of the space.

I' that of E'. All of which is in perfect agreement with the original syllogistic meaning and De Morgan's own views.

To illustrate this circumstance, we propose to adopt our usual notation for negation '¬' to symbolize this contradictory version of denial and rewrite the expressions in fig. 7 accordingly. In this way, the expression of a general duality for De Morgan's propositional forms becomes finally apparent. Thus, in the case of complex or conjunctive propositions, we have:

Dˌ	=	Aˌ	*and*	¬A'	|	¬Dˌ	=	A'	*or*	¬Aˌ
D	=	Aˌ	*and*	A'	|	¬D	=	¬A'	*or*	¬Aˌ
D'	=	A'	*and*	¬Aˌ	|	¬D'	=	Aˌ	*or*	¬A'
Cˌ	=	Eˌ	*and*	¬E'	|	¬Cˌ	=	E'	*or*	¬Eˌ
C	=	Eˌ	*and*	E'	|	¬C	=	¬E'	*or*	¬Eˌ
C'	=	E'	*and*	¬Eˌ	|	¬C'	=	Eˌ	*or*	¬E'

Likewise, in the case of simple or disjunctive propositions:[25]

Aˌ	=	Dˌ	*or*	D	|	¬Aˌ	=	¬Dˌ	*and*	¬D
A'	=	D'	*or*	D	|	¬A'	=	¬D'	*and*	¬D
Eˌ	=	Cˌ	*or*	C	|	¬Eˌ	=	¬Cˌ	*and*	¬C
E'	=	C'	*or*	C	|	¬E'	=	¬C'	*and*	¬C

In this way, we can see that De Morgan succeeds in embedding logical propositions in a whole new framework where their interplay, and even their very definition, are determined by a dual relation between conjunction and disjunction, under the action of an opposition understood as a contradictory version of denial. With this notion of contradictory denial, we find the third and last

[25]We have skipped the original (unaccentuated) propositional modes A, E, I, and O from syllogistics, which De Morgan had reintroduced in fig. 7 probably only for the sake of symmetry, revealing, if anything, their inadequacy within the new logical framework.

form of logical negation in De Morgan's system. Like the other two, this one can be interpreted by means of the simple mechanism of class complementation, acting here over the class characterizing the third layer in his edifice, namely that of propositional forms, corresponding in our contemporary reconstruction to $\mathcal{P}(N \times U)^*$.

4 De Morgan's De Morgan's Laws

After having examined with some detail the multiple senses of negation, conjunction, and disjunction in De Morgan's early logical work, we are now ready to address the place and meaning of his own statement of the famous duality principles.

The table in fig. 7 not only shows that De Morgan was widely attentive to dual properties but also that the latter constitute a central aspect of his system's logical power.[26] However, that table also reveals a remarkable fact: while previous extensions were accompanied with original notations capturing the objects and properties newly introduced (e.g., contrariness, contranominality, complex propositions, etc.), the operations upon which propositional duality is based, namely contradictory denial and propositional conjunction and disjunction, are only expressed in natural language ("denies", "affirms", "and", "or"). One could then expect that, from there, De Morgan would proceed to capture those operations with specific notations, granting them the status of actual propositional connectives, therefore leading the nascent formal logic into the direction of propositional logic as we know it. Expressed in the notation of such connectives, the dual properties derived at the level of propositions would then count as exact statements of what we now know as De Morgan's laws.[27]

Yet, *Formal Logic* follows an entirely different path. Instead of developing a new propositional logic based on his algebraic reconstruction of the syllogistic setting, De Morgan will redirect his attention back to names, and the principles of their interaction with regard to propositional forms. The statement of his duality principles belongs to this somewhat surprising orientation.

The context is given by the presentation of a "new view" on the nature of propositions, preparing the path to what De Morgan foresees as a new type of inference. Such a view stands on two related features. First, the introduction of *compound names*. So far, only simple names, like 'X' or 'Y', were considered in his system. Following a traditional view, any relation between names could only occur at a propositional level. However, De Morgan will propose to consider compound names, such as 'wild animal', as "the name of all things to which both the names *wild* and *animal* apply" (De Morgan, 2014, 105). Second, associated with compound names, De Morgan proposed an interpretation of names in terms of possibility and impossibility, "according as the thing to which it applies can be found or not" (De Morgan, 2014, 105).

Given the class semantics underlying De Morgan's construction, one would expect his introduction of nominal compositionality to be guided by the princi-

[26]Indeed, in the remaining pages, up to the treatment of syllogistic inference, De Morgan will continue to explore different logical symmetries and dualities based on those properties "to illustrate the want of the extension of the doctrine of propositions made in this chapter, and also the completeness of it" (De Morgan, 2014, 72).

[27]For a comprehensive view on the evolution of duality in logic during the 19th century, see Schlimm's contribution to this volume.

ples of class relations, typically intersection, much in the way in which contrary names were introduced through complementation, even if only in an informal way. However, while the definition of compound names coincides with that of class intersection, their operative aspects will rely on the purely symbolical means laid out for propositional forms, which again suggests the preeminence of symbolical operations over class semantics. Thus, De Morgan proposes the following notation:

> X and Y being two names, the compound name may be represented by XY when possible, and by XY) when impossible. This does not alter the meaning of our symbol XY, as hitherto used: as yet it has been 'there are Xs which are Ys' and now it is 'XY, the name of that which is both X and Y, is the name of some thing or things;' and these two are the same in meaning, so far as their use in inference is concerned. Nor need XY), as just defined, be treated as a departure from, otherwise than as an extension of, the use of X)Y. In X)Y, we assert that X is something, namely Y: in X) we assert that X is *nothing whatever*. (De Morgan, 2014, 106)

In this way, De Morgan can capture at the nominal level relations so far only expressible at a propositional one. The particular affirmative *propositional form* $\mathrm{I}_{\imath}$, symbolized XY (where concatenation is to be understood as a copulative symbol such as ')', ':' and '.', in line with his notation for propositions) can now be represented as a *name*, namely the compound XY, which has no propositional import.[28] Moreover, the interpretation of ')' as a symbol for impossibility allows the expression of all propositional forms by means of compound names:

> The proposition 'Every X is Y' asserts that Xy is the name of nothing, or X)Y=Xy). Similarly 'No X is Y' asserts that XY is the name of nothing, or X.Y=XY). But 'Some Xs are Ys' and 'Some Xs are not Ys' merely assert the possibility of the names XY and Xy.

Once again, in light of this reduction of propositional properties to a purely nominal dimension where elementary class principles hold, one would be tempted to conclude that De Morgan is leading logic into the path of modern propositional calculus as we know it, this time with the symbol ')' as the mark of falsity and its absence implicitly referring to truth.

Yet, once again, we would be wrong. For, despite the reduction of some of its traditional properties to the elementary mechanisms of names, the propositional level is not entirely discarded. The domain of propositional forms continues to be where inference takes place. In particular, even with its new meaning, copulative symbols like ')' continue to relate two nominal entities, the one to its right to the one to its left. To such an extent that it is precisely this requirement that forced De Morgan to introduce the symbol 'u', despite its uncertain origin:[29]

> The proper notation, however, for indicating that the name X has no application, is X)u, u being the contrary of U, which last includes everything in the universe spoken of; so that u may denote nonexistence. (De Morgan, 2014, 106)

[28] De Morgan disregards the risk of ambiguity, elaborating on the equivalence between both in (De Morgan, 2014, 116), where he suggests that compound names can be denoted with a hyphen, e.g., X-Y, whenever disambiguation is needed.

[29] See footnote 11.

However, capable of relating compound names, copulative symbols are now invested with new combinatorial capacities, requiring careful association with the principles of nominal interaction. In particular, transposition of terms from one side to the other of the copula will become possible under certain conditions. In the privileged case of ')', such transposition entails the transformation of a name into its contrary, as already suggested for Y in De Morgan's example given above: X)Y=Xy). If this new view on the nature of propositions is to be made systematic to support new inferential principles, then operations like these need to be precisely characterized.

This is the reason motivating De Morgan's statements of the duality principles. Required for a systematic transposition of compound names in a propositional form, the first of De Morgan's statements aims at defining the contrary of compound names. Only now the disjunctive composition is introduced accordingly:

> P, Q, R, being certain names, if we wish to give a name to everything which is all three, we may join them thus, PQR: if we wish to give a name to every thing which is either of the three (one or more of them) we may write P,Q,R: if we want to signify any thing that is either both P and Q, or R, we have PQ,R. The contrary of PQR is p,q,r; that of P,Q,R is pqr; that of PQ,R is (p,q)r: in contraries, conjunction and disjunction change places. (De Morgan, 2014, 115-116)

Once such dual principles are laid down for names, De Morgan proceeds, consistent with the structure of his system, "to extend this idea and notation relative to propositions of complex terms" (De Morgan, 2014, 118). This is precisely what his second statement does:

> The contrary of PQ, is p,q; that of P,Q is pq. *Not both* is either not one or not the other, or not either. *Not either P nor Q* (which we might denote by :P,Q or .P,Q) is logically '*not P and not Q*' or pq: and this is then the contrary of P,Q. (De Morgan, 2014, 118)

We can now see how both statements, practically indistinguishable from a contemporary standpoint, clearly differ under the light of his own system when sufficiently elucidated: while the first one concerns (compound) names, the second one concerns propositions.[30] It thus appears that the existence of two seemingly equivalent statements of the dual principles is actually the expression of a structuring distinction in De Morgan's system, namely that of the nominal and the propositional levels.

As we suggested, the reason for the irreducibility of the propositional form's domain to that of names is that the former holds the key to inferential principles, which, in the absence of a modern theory of classes, can only be represented through propositional forms in De Morgan's system (such as $\mathrm{A}_{\shortmid}$, $\mathrm{D}_{\shortmid}$ or D, for different forms of inclusion or identity of classes in U). If De Morgan wanted to erase the distinction between both dimensions entirely, he could engage in the development of a complete theory of classes providing such principles as

[30] As reflected by the presence of the copulative symbols ':' and '.', not to be mistaken for simple punctuation marks.

part of an external semantics. Interestingly, his approach points in a different direction. Rather than reducing the inferential properties of propositions to the interaction of complex names, he finds in the latter the source of new inferential principles with which to enrich the former.

In particular, the relation between compound names and each of their components is, for De Morgan, that of an "absolute identity", meaning that it only holds by definition, "for by the name PQ we signify nothing but what has right to both names [P and Q]." (De Morgan, 2014, 117). Now, even if only definitional, De Morgan perceives in this absolute identity a deductive principle, which is implicit in the informal expression he provides of that identity, namely: "Whatever has right to the name P, and also to the name Q, has right to the compound name PQ" (De Morgan, 2014, 117). Yet, while those sentences exhibit a deductive procedure, the latter cannot be captured by syllogistic means. In De Morgan's terms: "X)P+X)Q=X)PQ is not a syllogism, nor even an inference, but only the assertion of our right to use at our pleasure either one of two ways of saying the same thing instead of the other." (De Morgan, 2014, 117). As a piece evidence, De Morgan advances that, even if, given Y)PQ, both propositions Y)P and Y)Q can be derived, no syllogistic means allows us to infer X)Y from X)P and X)Q. And as if the nominal nature of the problem was not clear enough, he exclaims: "We might as well attempt to syllogize into the result, that a person who sells the meat he has killed is a butcher" (De Morgan, 2014, 117).

The logic of propositions and that of names obey then different deductive principles. While the former is subject to the rules of the syllogism, the latter follows those of the interaction of names, guided by the duality of compositional processes, grounded on the class semantics upon which De Morgan has built his system. At the point of contact of those two levels, compound names are the key to their articulation. Introduced, on the one hand, as the result of an extension of the propositional structure, they hold the power of translating back and forth between syllogistic inference and the language of names. Observing, on the other, the dual dynamics resulting from their class semantics, they involve a deductive principle irreducible to the classical means of syllogistics, even in their renewed form.

Accordingly, De Morgan's logical "new view" will hinge upon the possibility of investing with a logical status the nominal deductive power of compound names. This is what he aims at by introducing what he calls the "conjunctive postulate", that is, the "absolute, *less than inferential* (so to speak) identity of X)P + X)Q and X)PQ". Once this rule is introduced as a pure principle of "abstract expression", then "all other propositions of the kind, however simple, may be made deductions" (De Morgan, 2014, 118). In particular, De Morgan shows, without entirely dissimulating a triumphal tone, that the dual of the conjunctive postulate—i.e., the "disjunctive postulate": P)R + Q)R = P,Q)R—can be syllogistically derived. Indeed, we have that P)R + Q)R = r)p + r)q = r)pq = P,Q)R (De Morgan, 2014, 118).

In this way, conjunctive and disjunctive names are raised to the propositional level, and new mechanisms of propositional logic become available. However, De Morgan's proof of the disjunctive postulate clearly shows that such mechanisms rely upon the "less than inferential" rules of abstract expression with which he has premised his extension of the principles of syllogistic inference. In particular, other than the conjunctive postulate, his proof makes use of transposition of terms and nominal duality.

Once the conjunctive postulate has been laid down, De Morgan will continue, in the remaining pages of this chapter, to tackle the behavior of complex names within the propositional structure by establishing new inferential rules. In particular, he will determine in which cases components of complex names may be legitimately "rejected", given their propositional context. For instance, "any disjunctive element may be rejected from a universal term, and any conjunctive element from a particular one. Thus P)QR gives P)Q and P,Q)R gives P)R" (De Morgan, 2014, 119). In the same spirit, De Morgan elaborates a detailed analysis of the admissible rules of transposition (i.e., the "change from one member of the proposition to the other" (De Morgan, 2014, 120)), whose general principle had been introduced with the propositional interpretation of compound names. After considering all the possible combinations of conjunctive and disjunctive compound names within the propositional structure and evaluating the legitimate transpositions between their terms, De Morgan suggests that those rules can all be reduced to the ones established for the universal affirmative. Moreover, in this last case, only the form where conjunctive name is coupled with a disjunctive one (i.e., XY)P,Q) allows the transposition of any of their individual components, by changing the latter into their contraries (cf. De Morgan, 2014, 121).

Armed with these new inferential tools, De Morgan can finally address syllogistic inference anew and show, for instance, that "the ordinary disjunctive and dilemmatic forms are really common syllogisms with complex terms, reducible to ordinary syllogisms by invention of names" (De Morgan, 2014, 122). The first example he provides is illustrative of the mechanisms of the new framework: the syllogism 'Every S is either P, Q, R; no P is S; no Q is S; therefore, every S is R' can be now established by first transposing P and Q in S)P,Q,R to obtain Spq)R, then rewriting S.P and S.Q as S)p and S)q (following fig. 3) and applying the conjunctive postulate to obtain S)pq. Applying once again that postulate with respect to S)S (the use of which is "perfectly legitimate" (De Morgan, 2014, 122)) we obtain S)pqS which together with Spq)R gives the classical syllogism: S)pqS + pqS)R = S)R.[31]

Although fairly convoluted, De Morgan's approach is full of original ideas concerning the way to handle inference within a symbolic system without recourse to an external semantics.[32] At the heart of it lie his duality principles, less as laws derived from a clearly defined system of logical rules than as a structuring mechanism motivating the articulation of the different pieces of his edifice, whose construction has been implicitly or explicitly guided by dual properties.

Conclusion

In the present paper, we have attempted to contribute to the history of duality by providing an analytical insight into the place occupied by duality in the

[31] De Morgan assumes the rearranging of terms from Spq)R to pqS)R.

[32] We cannot resist here the temptation to evoke the mechanisms of Gentzen's sequent calculus at the origin of the proof-theoretical approach to logic (cf. Gentzen, 1969 (1935), with which De Morgan's elaborations in this sense bear surprising resemblance. A detailed analysis beyond this external resemblance falls, unfortunately, out of the scope of the present paper.

emergence of mathematized systems of logic at the turn of the 19th century. More precisely, we examined De Morgan's early logical work to provide a detailed analysis of the mechanisms motivating not only the extensive presence of dual properties in his system of logic but also what can arguably be taken as the first explicit statement in his work of the famous duality principles named after him. We thus presented a detailed analysis of the form that take, in De Morgan's early system, the two main features involved in the abstract characterization of dual mechanisms, namely the interaction between different forms of negation and the relation between conjunctive and disjunctive properties.

From this analysis, it appeared that the recourse to an embryonic practice of class complementation provided De Morgan with a powerful semantics for an original treatment of logical negation. However, the latter is far from reducible to a simple form of propositional negation like that of our modern propositional calculus. In contrast, we found that, in De Morgan's *Formal Logic*, logical negation adopts three different main forms, corresponding to the stratified structure of his system, determined by objects of increasing complexity: *names*, defining classes of individual objects; *propositional relations*, prescribing relations between names and objects; and *propositional forms*, conceived as classes of those relations. At each of those stages, a particular form of opposition is at play, guided by what we proposed to interpret as a practice of class complementation over a different class of objects: U, $N \times U$, and $\mathcal{P}(N \times U)^*$, respectively. Each of those cases of class complementation provides the elementary semantic requirements for a different form of logical negation: *contrariness*, *contranominality*, and *contradictory denial*.

As a consequence of the embryonic class theoretical characteristics of each of those three layers in De Morgan's system, we saw that the interaction between their respective negations motivated the emergence of multiple forms of symmetry between objects and properties, capable of exhibiting dual properties. That stratified structure permits then to elucidate why De Morgan's system can be characterized as "rather akin to duality", as Grattan-Guinness suggested. In particular, we showed that the relation between contrariness and contranominality behaved like that of an internal to an external negation, and that the general predisposition to duality encouraged the identification of contradictory denial as yet another form of negation for propositional forms, mirroring syllogistics contradiction with its traditional dual behavior with respect to quantity and quality.

As for conjunctive and disjunctive properties, we showed that conjunction and disjunction in De Morgan's early system were not directly rooted in the mechanisms of a unified elementary class semantics informing compositional principles between propositions, as for contemporary logical connectives. Instead, De Morgan considered them as intrinsic properties of his system's logical objects, associated with the symmetries induced by the multiple forms of negation. This conception informed, in particular, an original conception of the nature of propositional forms where, although elementary, propositions are not atomic, for they are always internally articulated either conjunctively or disjunctively. In this novel approach, propositional forms can be seen to represent class relations such as identity, inclusion, or disjunction for classes of individuals in the universe of discourse. As such, conjunction and disjunction appear, in De Morgan's early logical view, as two inverted perspectives the internal structure of necessarily articulated propositions provide on those class relations.

More generally, the logical edifice where those original notions of negation, conjunction, and disjunction unfold can be seen as the expression of the laborious construction of a class semantics for the received propositional setting of syllogistic logic (construction of which Gergonne counts as a discreet pioneer). However, our analysis repeatedly showed that any attempt to project upon the resulting system the image of a propositional calculus as we know it, where the duality of the so-called 'De Morgan's laws' follow from definitions, should be regarded with suspicion. For not only do the objects and properties stemming from De Morgan's original views differ radically from those resulting from a simple correspondence between class-theoretical relations and logical operations, but De Morgan systematically misses the opportunities to engage in that direction. In particular, class intersection and union are never considered as primitives in his early system and either implicitly guide some of his constructions or find a derived representation through propositional forms.

Accordingly, the place granted to duality in De Morgan's approach also differs from our contemporary understanding of propositional logic in a significant way. De Morgan's stratified semantics of classes presented duality properties at two levels at least. At the propositional level, duality emerges as a consequence of the partial order induced by simple and complex propositions, exchanging between their disjunctive and conjunctive articulation through contradictory denial as complementation over $\mathcal{P}(N \times U)^*$. At the nominal level, a dual behavior results from the partial order induced upon U by names through the relation R, exchanging, through name contrariness as complementation over U, the conjunctive and disjunctive principles of compound names.

As a consequence of these two privileged forms of duality within the stratified class semantics, De Morgan's resulting system for logic is essentially twofold, consisting of a propositional layer where propositional forms interact through the renewed principles of syllogistic inference (e.g., $\neg\,(\mathrm{D}_{,}\ \textit{or}\ \mathrm{D}) = \neg\mathrm{D}_{,}\ \textit{and}\ \neg\mathrm{D}$; cf. p. 26); and an underlying symbolical yet sub-propositional layer of names interacting through compositional principles of "abstract expression" (e.g., p,q,r as the contrary of PQR; cf. p. 29). In such a configuration, the formal interaction between those two layers becomes the key to the new logical perspective De Morgan intends to inaugurate.

It was at this precise point that De Morgan introduced his explicit statement of his famous laws, which confirms the centrality granted to duality in his reconstruction of logic. Owing to the twofold character of his system, two statements of those principles can be found at this stage: a nominal version and its propositional counterpart. Significantly, the nominal version of the duality principles constituted a necessary condition for extending the sub-propositional "less than inferential" power of the conjunctive postulate to other modes of inference.

De Morgan's elaboration of his new system was unquestionably clumsy and lacked a clear and explicit systematicity. Cases and exceptions multiplied each time conditions for rules were drawn, without foreseeable simplification or derivation of a more general principle. In addition, most distinctions were stated without further use of them. All of which bears testimony to the fact that the sought articulation between the nominal and the propositional dimensions of logic was not properly accomplished. In particular, the formal connection between the duality of complex names and that induced by conjunctive and disjunctive propositions was never fully addressed as such. After his seminal *Formal Logic*, De Morgan focused increasingly on the logic of names or "ony-

X)·(Y. No X is Y; everything either x or y: X and Y have no common part: but, if not complements, have common wholes. Every individual is in some of the parts either of x or of y: and is either not in some whole of X, or not in some whole of Y. That is, no junction of a new attribute selects any part of one out of the other: everything wants some attribute of one or the other.	X(·)Y. No x is y: everything either X or Y: X and Y have no common[2] whole: but, if not externals, have common parts. Every individual is in all the wholes either of X or of Y, and is either not any part of x, or not any part of y. That is, no dismissal of an existing attribute makes any whole of one a whole of the other: everything has all the attributes of one or the other.

Figure 9: Example of correspondence of propositions in a two-column layout. (De Morgan, 1863, 476)

matic system", as he called it (cf. De Morgan (2019, 119-120) and (1863)). He presented nominal duality as stemming from the inverse relationship between extension and intension, naturally induced by the relation R upon the universe U and the class N of names in our contemporary reconstruction. Renaming the disjunction of names as *aggregation* and their conjunction as *composition*, De Morgan rephrased his duality principles accordingly, in the way referred at the beginning of this paper (p. 2). Interestingly, the development of this logical duality continued producing "correlations" between different kinds of propositions, which De Morgan presented in the typical two-column layout (fig. 9).

However, despite the multiple difficulties associated with his initial framework, De Morgan's reluctance to clearly go down the path leading to modern propositional logic is revealing of the alternative program he devised for formal logic. Such a program was organized around the simultaneous existence of two different layers within formal systems: a purely nominal one, governed by rules of abstract expression, and a propositional one, where necessarily articulated propositions support logical inference owing to the mechanisms of their internal structure. Under such conditions, logical operations, such as negation, conjunction, and disjunction, are not primitive components of the system but emergent global properties resulting from the interaction of expressive components, such as names, within a propositional structure. Hence the De Morgan's insistence on the adjectival form of those operations ("conjunctive", "disjunctive"); hence, also, the privileged place granted to a semantics of negation as the abstract operation exchanging conjunctive behaviors into disjunctive ones and vice-versa. All of which speaks, as we have suggested, for the design to handle logical inference in a symbolic system without recourse to an external semantics, rather than to develop a truth-functional semantics for propositional connectives.

In his introduction to the selected manuscripts of George Boole, Grattan-Guinness connects the emergence of Boole's work to an alternative tradition to classical syllogistic thought, stemming from Locke, which "[s]howing more sympathy to the role of language in logic than had normally obtained among the syllogists, [...] laid emphasis on signs as keys to logical knowledge" (Grattan-Guinness, 1997, xxvi). Grattan-Guinness then referred to the word *semiotics*, introduced by Locke to characterize this "sign tradition". Speaking of De Morgan's symbolic practices in his logical work, Grattan-Guinness also affirms that "[h]is status in the history of semiotics should be raised" (Grattan-Guinness, 2000, 36). We believe that the analyses proposed in the present paper contribute to shedding some light on that semiotic origin of modern logic and the fundamental role duality played in it.

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